

Dirac equation in Kerr-Taub NUT spacetime

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Abstract

We study Dirac equation in Kerr-Taub-NUT spacetime. We use Boyer-Lindquist coordinates and separate the resulting equations into radial and angular parts. We get some exact analytical solutions of the angular equations for some special cases. We also obtain the radial wave equations with an effective potential. Finally we discuss the potentials by plotting them as a function of radial distance in a physically acceptable region.

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1 Introduction

Kerr-Taub-NUT (Newman-Uni-Tamburino) spacetime is the generalized solution of the vacuum Einstein field equations in general relativity [1]. The solution is obtained by introducing an extra non-trivial magnetic mass parameter which is also called "gravitomagnetic monopole moment" (NUT charge). Kerr-Taub-NUT solution describes the spacetime of a localized stationary and axially symmetric object [1]. The solution contains three physical parameters: The gravitational mass, which is also called gravitoelectric charge; the gravitomagnetic mass that is also identified as the NUT charge; the rotation parameter that is the angular speed per unit mass. The NUT charge produces an asymptotically non-flat spacetime in contrast to Kerr geometry that is asymptotically flat [2]. Although the Kerr-Taub-NUT spacetime has no curvature singularities, there exist conical singularities on the axis of symmetry [3]. One can get rid off conical singularities by taking a periodicity condition over the time coordinate. But, this leads to the emergence of closed time-like curves in the spacetime. It means that, in contrast to Kerr solution interpreted as a regular rotating black hole, the Kerr-Taub-NUT solution cannot be identified as a regular black hole solution due to its singularity structure. An alternative physical interpretation of Kerr-Taub-NUT spacetime can be found in [4] where the NUT metric is interpreted as a semi-infinite massless source of angular momentum. Despite the fact that Kerr-Taub-NUT solution has some unpleasing properties, it is vastly studied for exploring various physical phenomena in general relativity due to its asymptotically non-flat spacetime structure [5, 6, 7].

In the present paper, we study the Dirac equation in Kerr-Taub-NUT spacetime. Dirac equation has been extensively examined in various gravitational spacetimes including Schwarzschild geometry [8], Kerr-spacetime [9, 10, 11, 12], 4-dimensional constant-curvature black hole spacetime [13], rotating Bertotti-Robinson geometry [14], 4-dimensional Nutku helicoid spacetime [15] and open universe geometry [16]. In all these spacetimes except the Kerr geometry, exact solutions of Dirac equation have been found for all possible values of physical parameters. However in the rotating Kerr spacetime, exact solutions of Dirac equation have been obtained only for some special values of the parameters [11, 12]. In [11], the series solution of angular Dirac equation have been given while in [12], angular solutions have been presented by using spectral decomposition method in which the angular wave functions are expanded in terms of spheroidal harmonics. By this method, a three-term

recursion relation is achieved and eigenvalues of the angular equations are solved.

On the other hand, in all these works, the separability of the Dirac equation has been accomplished. Separability was first discovered in Hamilton-Jacobi and relativistic wave equations by Carter [17]. Later on it has been discussed in higher dimensional spacetimes in the works [18, 19, 20]. Separability also implies the existence of integrals of motion associated with the second order symmetric Stäckel-Killing tensors [19, 20]. On the other hand, the separability of the Dirac equation was first discovered by Chandrasekhar [9, 10] and later on it was extended to Kerr-Newman [21] and AdS-Kerr-Taub-NUT spacetimes [22].

In this work, we obtain the set of equations by employing an axially symmetric ansatz for the Dirac spinor. Equations obtained are separated into radial and angular parts with appropriate substitutions of spinor fields. We try to solve angular equations exactly. But unfortunately, we are unable to get exact analytical solutions to general angular equations for all physical parameters. Under some restrictions implemented on the separation constant, we present some exact solutions of the equations with and without gravitomagnetic mass and rotation parameters. Indeed, they can be solved exactly in terms of hypergeometric functions for the cases where the mass of the Dirac particle is equal to or twice the frequency of the spinor wave function. In the final part, radial equations are discussed. With some transformations on the dependent and independent field variables, wave equations with an effective potential barrier are obtained. To understand the physical behavior of the potentials, they are plotted with changing frequency and gravitomagnetic mass parameter in the physically acceptable regions.

Organization of the paper is as follows: In section 2, we present the general form of Dirac equation in exterior forms. In section 3, we obtain Dirac equation in Kerr-Taub-NUT spacetime. In the subsections, we discuss the separability of the equations, obtain the angular and radial equations. Next, we find some exact analytical solutions of the angular equations. Finally, we study the radial wave equations and examine the behavior of the potential barriers that come out in the transformed radial equations. We end up with some comments and conclusions.

2 Dirac equation in 4-dimensional spacetime

We consider a 4-dimensional spacetime manifold M equipped with a Lorentzian metric g with signature $(-, +, +, +)$ and a metric compatible connection ∇ . We assume that our spacetime manifold has a spin structure group $Spin_+(3, 1)$. It is known that the fundamental group of Lorentzian group $SO_+(3, 1)$ is \mathbb{Z}_2 so that it has a universal covering group of $Spin_+(3, 1)$ that is the multiplicative subgroup of complex Clifford algebra $\mathbb{C}l_{3,1}$.

In exterior forms, the Dirac equation can be written as [23]

$$* \gamma \wedge D\psi + \mu \psi * 1 = 0, \quad (2.1)$$

where γ is $\mathbb{C}l_{3,1}$ -valued 1-form $\gamma = \gamma^a e_a$. We choose the units such that $c = 1$ and $\hbar = 1$. Here $*$ denotes Hodge-star operator and μ is the mass of the particle. $\{e_a\}$'s are the orthonormal co-frame 1-forms such that the metric $g = \eta_{ab} e^a \otimes e^b$. ψ represents \mathbb{C}^4 -valued Dirac spinor whose covariant exterior derivative can be written as

$$D\psi = d\psi + \frac{1}{2} \sigma^{ab} \omega_{ab} \psi, \quad (2.2)$$

where $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ and $\{\gamma^a\}$'s satisfy the relations

$$\{\gamma^a, \gamma^b\} = (\gamma^a \gamma^b + \gamma^b \gamma^a) = 2\eta^{ab} I_{4 \times 4}.$$

ω_{ab} are the connection 1-forms that satisfy Cartan structure equations

$$de^a + \omega^a{}_b \wedge e^b = T^a$$

where T^a denotes torsion 2-form and metric compatibility implies that $w_{ab} = -w_{ba}$.

Since $\mathbb{C}l_{3,1}$ is isomorphic to $\mathcal{M}_4(\mathbb{C})$ that is the set of 4×4 complex matrices, we can choose the representation

$$\begin{aligned} \gamma^0 &= i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^1 = i \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \\ \gamma^2 &= i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = i \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}, \end{aligned}$$

where σ^i are the Pauli spin matrices and I is the 2×2 identity matrix.

3 Dirac Equation in Kerr-Taub-NUT Spacetime

In this section, we examine the Dirac equation in Kerr-Taub-Nut spacetime. In Boyer-Lindquist coordinates, Kerr-Taub-NUT spacetime can be described by the metric with asymptotically non-flat structure,

$$g = -\frac{\Delta}{\Sigma}(dt - \chi d\varphi)^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma} (adt - (r^2 + \ell^2 + a^2)d\varphi)^2 \quad (3.1)$$

where

$$\Sigma = r^2 + (\ell + a \cos \theta)^2, \quad \Delta = r^2 - 2Mr + a^2 - \ell^2$$

and

$$\chi = a \sin^2 \theta - 2\ell \cos \theta.$$

Here, M is a parameter related to physical mass of the gravitational source. a is associated with its angular momentum per unit mass and ℓ denotes gravitomagnetic monopole moment of the source. For the metric (3.1), we choose the co-frame 1-forms

$$\begin{aligned} e^0 &= \left(\frac{\Delta}{\Sigma} \right)^{1/2} (dt - \chi d\varphi), & e^1 &= \left(\frac{\Sigma}{\Delta} \right)^{1/2} dr, \\ e^2 &= \Sigma^{1/2} d\theta, & e^3 &= \frac{\sin \theta}{\Sigma^{1/2}} (adt - (r^2 + \ell^2 + a^2)d\varphi). \end{aligned} \quad (3.2)$$

We consider that the spacetime is Levi-Civita (torsion-free) such that connection 1-forms $\omega^a{}_b$ can be determined from the equation

$$de^a + \omega^a{}_b \wedge e^b = 0$$

which has a unique solution

$$\omega^a{}_b = \frac{1}{2} (e^c (\iota^a \iota_b de_c) + \iota_b de^a - \iota^a de_b). \quad (3.3)$$

Here $\iota_a = \iota_{X_a}$ are inner-product operators that satisfy $\iota_b e^a = \delta_b^a$. From equation (3.3), we can determine connection 1-forms $\omega^a{}_b$:

$$\begin{aligned} \omega^0{}_1 &= A_2 dt + A_3 d\varphi, & \omega^0{}_2 &= B_3 d\varphi, \\ \omega^0{}_3 &= C_1 dr + C_2 d\theta, & \omega^1{}_2 &= E_1 dr + E_2 d\theta, \\ \omega^1{}_3 &= G_3 d\varphi, & \omega^2{}_3 &= K_2 dt + K_3 d\varphi, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned}
A_2 &= \frac{Mr^2 + 2\ell^2 r + 2\ell ar \cos \theta - M(\ell + a \cos \theta)^2}{\Sigma^2} \\
A_3 &= \frac{\chi \{(m-r)\Sigma - 2\ell^2 r - 2Mr^2\} - 2\ell r \cos \theta (r^2 + \ell^2 + a^2)}{\Sigma^2} \\
B_3 &= -\frac{\Delta^{1/2} \sin \theta (\ell + a \cos \theta)}{\Sigma}, \quad C_1 = \frac{ar \sin \theta}{\Sigma \Delta^{1/2}}, \quad C_2 = -\frac{\Delta^{1/2}}{\Sigma} (\ell + a \cos \theta), \\
E_1 &= -\frac{a \sin \theta (\ell + a \cos \theta)}{\Delta^{1/2} \Sigma}, \quad E_2 = -\frac{r \Delta^{1/2}}{\Sigma}, \quad G_3 = \frac{r \sin \theta \Delta^{1/2}}{\Sigma}, \quad (3.5) \\
K_2 &= \frac{\ell r^2 - (\ell + a \cos \theta)(2Mr + \ell^2 + a\ell \cos \theta)}{\Sigma^2}, \\
K_3 &= \frac{\cos \theta (r^2 + \ell^2 + a^2)(\Sigma + 2\ell(\ell + a \cos \theta)) + \chi(2Mr + 2\ell^2)(\ell + a \cos \theta)}{\Sigma^2}.
\end{aligned}$$

Since the space-time is axially symmetric, we can take

$$\psi = e^{-i\omega t} e^{im\varphi} \begin{pmatrix} \psi_1(r, \theta) \\ \psi_2(r, \theta) \\ \psi_3(r, \theta) \\ \psi_4(r, \theta) \end{pmatrix}, \quad (3.6)$$

where m denotes azimuthal quantum number. Then we substitute (3.4), (3.5) and (3.6) into Dirac equation (2.1) and obtain the following equations:

$$\left(\frac{\omega\alpha_1 + m\alpha_2}{\Delta^{1/2} \sin \theta} \right) \psi_3 - \frac{i\Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_4}{\partial r} - \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_4}{\partial \theta} + \frac{i}{2}(\delta_2 + \delta_4)\psi_4 + \frac{1}{2}(\delta_1 - \delta_3)\psi_4 + \mu\psi_1 = 0, \quad (3.7)$$

$$\left(\frac{-\omega\alpha_3 + m\alpha_4}{\Delta^{1/2} \sin \theta} \right) \psi_4 - \frac{i\Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_3}{\partial r} + \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_3}{\partial \theta} + \frac{i}{2}(\delta_4 - \delta_2)\psi_3 + \frac{1}{2}(\delta_1 + \delta_3)\psi_3 + \mu\psi_2 = 0, \quad (3.8)$$

$$\left(\frac{-\omega\alpha_3 + m\alpha_4}{\Delta^{1/2} \sin \theta}\right) \psi_1 + \frac{i\Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_2}{\partial r} + \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_2}{\partial \theta} + \frac{i}{2}(\delta_2 - \delta_4)\psi_2 + \frac{1}{2}(\delta_1 + \delta_3)\psi_2 + \mu\psi_3 = 0, \quad (3.9)$$

$$\left(\frac{\omega\alpha_1 + m\alpha_2}{\Delta^{1/2} \sin \theta}\right) \psi_2 + \frac{i\Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_1}{\partial r} - \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_1}{\partial \theta} - \frac{i}{2}(\delta_2 + \delta_4)\psi_1 + \frac{1}{2}(\delta_1 - \delta_3)\psi_1 + \mu\psi_4 = 0, \quad (3.10)$$

where

$$\alpha_1 = \frac{\chi\Delta^{1/2} - (r^2 + a^2 + \ell^2) \sin \theta}{\Sigma^{1/2}}, \quad \alpha_2 = \frac{a \sin \theta - \Delta^{1/2}}{\Sigma^{1/2}},$$

$$\alpha_3 = \frac{\chi\Delta^{1/2} + (r^2 + a^2 + \ell^2) \sin \theta}{\Sigma^{1/2}}, \quad \alpha_4 = \frac{a \sin \theta + \Delta^{1/2}}{\Sigma^{1/2}},$$

and

$$\delta_1 = -\frac{(l + a \cos \theta)}{\Sigma^{3/2}} \Delta^{1/2}, \quad \delta_2 = \frac{ar \sin \theta}{\Sigma^{3/2}},$$

$$\delta_3 = \frac{\cos \theta \Sigma - a \sin^2 \theta (l + a \cos \theta)}{\sin \theta \Sigma^{3/2}}, \quad \delta_4 = -\frac{(\Sigma(r - M) + r\Delta)}{\Delta^{1/2} \Sigma^{3/2}}.$$

3.1 Separability of the equations

Adding and subtracting equations (3.7), (3.8), (3.9), (3.10) and defining

$$\begin{aligned} F_1 &= \psi_1 + \psi_2 \\ F_2 &= \psi_2 - \psi_1 \\ F_3 &= \psi_3 + \psi_4 \\ F_4 &= \psi_4 - \psi_3, \end{aligned} \quad (3.11)$$

we obtain

$$\begin{aligned} &\frac{1}{\Sigma^{1/2}} \left\{ \frac{ma - \omega(r^2 + a^2 + \ell^2)}{\Delta^{1/2}} - \frac{i}{2} \frac{\Delta^{1/2}}{\Sigma} (r - i(\ell + a \cos \theta)) - \mathcal{D} \right\} F_3 \\ &+ \frac{1}{\Sigma^{1/2}} \left\{ \frac{m - \omega\chi}{\sin \theta} + \frac{i}{2} \frac{a \sin \theta}{\Sigma} (r - i(\ell + a \cos \theta)) - \mathcal{L} \right\} F_4 + \mu F_1 = 0, \end{aligned} \quad (3.12)$$

$$\begin{aligned} &\frac{1}{\Sigma^{1/2}} \left\{ \frac{-ma + \omega(r^2 + a^2 + \ell^2)}{\Delta^{1/2}} - \frac{i}{2} \frac{\Delta^{1/2}}{\Sigma} (r - i(\ell + a \cos \theta)) - \mathcal{D} \right\} F_4 \\ &+ \frac{1}{\Sigma^{1/2}} \left\{ \frac{-m + \omega\chi}{\sin \theta} + \frac{i}{2} \frac{a \sin \theta}{\Sigma} (r - i(\ell + a \cos \theta)) - \mathcal{L} \right\} F_3 - \mu F_2 = 0, \end{aligned} \quad (3.13)$$

$$\begin{aligned} & \frac{1}{\Sigma^{1/2}} \left\{ \frac{ma - \omega(r^2 + a^2 + \ell^2)}{\Delta^{1/2}} + \frac{i}{2} \frac{\Delta^{1/2}}{\Sigma} (r + i(\ell + a \cos \theta)) + \mathcal{D} \right\} F_1 \\ & + \frac{1}{\Sigma^{1/2}} \left\{ \frac{-m + \omega\chi}{\sin \theta} + \frac{i}{2} \frac{a \sin \theta}{\Sigma} (r + i(\ell + a \cos \theta)) + \mathcal{L} \right\} F_2 + \mu F_3 = 0, \end{aligned} \quad (3.14)$$

$$\begin{aligned} & \frac{1}{\Sigma^{1/2}} \left\{ \frac{-ma + \omega(r^2 + a^2 + \ell^2)}{\Delta^{1/2}} + \frac{i}{2} \frac{\Delta^{1/2}}{\Sigma} (r + i(\ell + a \cos \theta)) + \mathcal{D} \right\} F_2 \\ & + \frac{1}{\Sigma^{1/2}} \left\{ \frac{m - \omega\chi}{\sin \theta} + \frac{i}{2} \frac{a \sin \theta}{\Sigma} (r + i(\ell + a \cos \theta)) + \mathcal{L} \right\} F_1 - \mu F_4 = 0, \end{aligned} \quad (3.15)$$

where

$$\mathcal{D} = \frac{i}{2} \left(\frac{r - M}{\Delta^{1/2}} + \Delta^{1/2} \frac{\partial}{\partial r} \right), \quad \mathcal{L} = \frac{1}{2} \cot \theta + \frac{\partial}{\partial \theta}.$$

At this stage we redefine

$$\begin{aligned} \bar{F}_1 &= i(r - i(\ell + a \cos \theta))^{1/2} F_1, \\ \bar{F}_2 &= -i(r - i(\ell + a \cos \theta))^{1/2} F_2, \\ \bar{F}_3 &= (r + i(\ell + a \cos \theta))^{1/2} F_3, \\ \bar{F}_4 &= (r + i(\ell + a \cos \theta))^{1/2} F_4. \end{aligned} \quad (3.16)$$

Then we multiply equations (3.12) and (3.13) by $\Sigma^{1/2} (r - i(\ell + a \cos \theta))^{1/2}$ and equations (3.14) and (3.15) by $\Sigma^{1/2} (r + i(\ell + a \cos \theta))^{1/2}$ and we simplify resulting equations. We finally get

$$\begin{aligned} & \left\{ \frac{ma - \omega(r^2 + \ell^2 + a^2)}{\Delta^{1/2}} - \mathcal{D} \right\} \bar{F}_3 + \left\{ \frac{m - \omega\chi}{\sin \theta} - \mathcal{L} \right\} \bar{F}_4 \\ & - \mu(ir - (\ell + a \cos \theta)) \bar{F}_1 = 0, \end{aligned} \quad (3.17)$$

$$\begin{aligned} & \left\{ \frac{-m + \omega\chi}{\sin \theta} - \mathcal{L} \right\} \bar{F}_3 + \left\{ \frac{-ma + \omega(r^2 + \ell^2 + a^2)}{\Delta^{1/2}} - \mathcal{D} \right\} \bar{F}_4 \\ & - \mu(ir - (\ell + a \cos \theta)) \bar{F}_2 = 0, \end{aligned} \quad (3.18)$$

$$\begin{aligned} & \left\{ \frac{ma - \omega(r^2 + \ell^2 + a^2)}{\Delta^{1/2}} + \mathcal{D} \right\} \bar{F}_1 + \left\{ \frac{m - \omega\chi}{\sin \theta} - \mathcal{L} \right\} \bar{F}_2 \\ & + \mu(ir + (\ell + a \cos \theta)) \bar{F}_3 = 0, \end{aligned} \quad (3.19)$$

$$\left\{ \frac{m - \omega\chi}{\sin\theta} + \mathcal{L} \right\} \bar{F}_1 + \left\{ \frac{ma - \omega(r^2 + \ell^2 + a^2)}{\Delta^{1/2}} - \mathcal{D} \right\} \bar{F}_2 - \mu(ir + (\ell + a \cos\theta)) \bar{F}_4 = 0. \quad (3.20)$$

Equations (3.17), (3.18), (3.19) and (3.20) imply the separability ansatz

$$\begin{aligned} \bar{F}_1 &= R_1(r)S_1(\theta), \\ \bar{F}_2 &= R_2(r)S_2(\theta), \\ \bar{F}_3 &= R_2(r)S_1(\theta), \\ \bar{F}_4 &= R_1(r)S_2(\theta). \end{aligned} \quad (3.21)$$

With the ansatz above, equations take the following forms:

$$\begin{aligned} & \left[\left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D} \right\} R_2(r) - i\mu r R_1(r) \right] S_1(\theta) \\ & + \left[\left\{ \frac{m - \omega\chi}{\sin\theta} - \mathcal{L} \right\} S_2(\theta) + \mu(\ell + a \cos\theta) S_1(\theta) \right] R_1(r) = 0, \end{aligned} \quad (3.22)$$

$$\begin{aligned} & \left[\left\{ \frac{-m + \omega\chi}{\sin\theta} - \mathcal{L} \right\} S_1(\theta) + \mu(\ell + a \cos\theta) S_2(\theta) \right] R_2(r) \\ & + \left[\left\{ \frac{-ma + \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D} \right\} R_1(r) - i\mu r R_2(r) \right] S_2(\theta) = 0, \end{aligned} \quad (3.23)$$

$$\begin{aligned} & \left[\left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + \mathcal{D} \right\} R_1(r) + i\mu r R_2(r) \right] S_1(\theta) \\ & + \left[\left\{ \frac{m - \omega\chi}{\sin\theta} - \mathcal{L} \right\} S_2(\theta) + \mu(\ell + a \cos\theta) S_1(\theta) \right] R_2(r) = 0, \end{aligned} \quad (3.24)$$

$$\begin{aligned} & \left[\left\{ \frac{m - \omega\chi}{\sin\theta} + \mathcal{L} \right\} S_1(\theta) - \mu(\ell + a \cos\theta) S_2(\theta) \right] R_1(r) \\ & + \left[\left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D} \right\} R_2(r) - i\mu r R_1(r) \right] S_2(\theta) = 0. \end{aligned} \quad (3.25)$$

These equations further imply that

$$\lambda_1 R_1(r) = \left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D} \right\} R_2(r) - i\mu r R_1(r), \quad (3.26)$$

$$\lambda_2 R_2(r) = \left\{ \frac{-ma + \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D} \right\} R_1(r) - i\mu r R_2(r), \quad (3.27)$$

$$\lambda_3 R_2(r) = \left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + \mathcal{D} \right\} R_1(r) + i\mu r R_2(r), \quad (3.28)$$

$$\lambda_4 R_1(r) = \left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + \mathcal{D} \right\} R_2(r) - i\mu r R_1(r), \quad (3.29)$$

and

$$\lambda_1 S_1(\theta) = \left\{ \frac{-m + \omega\chi}{\sin \theta} + \mathcal{L} \right\} S_2(\theta) - \mu(\ell + a \cos \theta) S_1(\theta), \quad (3.30)$$

$$\lambda_2 S_2(\theta) = \left\{ \frac{m - \omega\chi}{\sin \theta} + \mathcal{L} \right\} S_1(\theta) - \mu(\ell + a \cos \theta) S_2(\theta), \quad (3.31)$$

$$\lambda_3 S_1(\theta) = \left\{ \frac{-m + \omega\chi}{\sin \theta} + \mathcal{L} \right\} S_2(\theta) - \mu(\ell + a \cos \theta) S_1(\theta), \quad (3.32)$$

$$\lambda_4 S_2(\theta) = \left\{ \frac{-m + \omega\chi}{\sin \theta} - \mathcal{L} \right\} S_1(\theta) + \mu(\ell + a \cos \theta) S_2(\theta). \quad (3.33)$$

For consistency of the equations (3.26)-(3.33), we choose $\lambda_1 = \lambda_3 = \lambda_4 = \lambda$ and $\lambda_2 = -\lambda$. Then we obtain following independent radial and angular equations:

$$\lambda R_1(r) = \left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D} \right\} R_2(r) - i\mu r R_1(r), \quad (3.34)$$

$$\lambda R_2(r) = \left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + \mathcal{D} \right\} R_1(r) + i\mu r R_2(r), \quad (3.35)$$

and

$$\lambda S_2(\theta) = \left\{ \frac{\omega\chi - m}{\sin \theta} - \mathcal{L} \right\} S_1(\theta) + \mu(\ell + a \cos \theta) S_2(\theta), \quad (3.36)$$

$$\lambda S_1(\theta) = \left\{ \frac{\omega\chi - m}{\sin \theta} + \mathcal{L} \right\} S_2(\theta) - \mu(\ell + a \cos \theta) S_1(\theta). \quad (3.37)$$

3.2 Angular equations

Angular equations (3.36) and (3.37) can be arranged as

$$\frac{dS_1}{d\theta} + \left\{ \left(\frac{1}{2} + 2\omega\ell \right) \cot \theta - a\omega \sin \theta + \frac{m}{\sin \theta} \right\} S_1(\theta) = (\mu(\ell + a \cos \theta) - \lambda) S_2(\theta), \quad (3.38)$$

and

$$\frac{dS_2}{d\theta} + \left\{ \left(\frac{1}{2} - 2\omega\ell \right) \cot \theta + a\omega \sin \theta - \frac{m}{\sin \theta} \right\} S_2(\theta) = (\mu(\ell + a \cos \theta) + \lambda) S_1(\theta). \quad (3.39)$$

By affecting the transformation

$$\begin{aligned} S_1 &= \cos \left(\frac{\theta}{2} \right) T_1 + \sin \left(\frac{\theta}{2} \right) T_2 \\ S_2 &= -\sin \left(\frac{\theta}{2} \right) T_1 + \cos \left(\frac{\theta}{2} \right) T_2, \end{aligned} \quad (3.40)$$

we get the following equations:

$$\begin{aligned} \frac{dT_1}{d\theta} + \left\{ \left(\frac{1}{2} + m \right) \cot \theta + \ell(\mu - 2\omega) \sin \theta + \frac{2\omega\ell}{\sin \theta} + (a\mu - a\omega) \sin \theta \cos \theta \right\} T_1 \\ + \left\{ \left(m + \lambda + \frac{1}{2} \right) + \ell(2\omega - \mu) \cos \theta - a\omega \sin^2 \theta - a\mu \cos^2 \theta \right\} T_2 = 0, \end{aligned} \quad (3.41)$$

$$\begin{aligned} \frac{dT_2}{d\theta} + \left\{ \left(\frac{1}{2} - m \right) \cot \theta - \ell(\mu - 2\omega) \sin \theta - \frac{2\omega\ell}{\sin \theta} + (a\mu - a\omega) \sin \theta \cos \theta \right\} T_2 \\ + \left\{ \left(m - \lambda - \frac{1}{2} \right) + \ell(2\omega - \mu) \cos \theta - a\omega \sin^2 \theta - a\mu \cos^2 \theta \right\} T_1 = 0, \end{aligned} \quad (3.42)$$

If we redefine $T_1 = T_+$ and $T_2 = T_-$, we see that (3.41) and (3.42) satisfy the following second order differential equations:

$$\frac{d^2 T_{\pm}}{d\theta^2} + M_{\pm} \frac{dT_{\pm}}{d\theta} + N_{\pm} T_{\pm} = 0, \quad (3.43)$$

where

$$M_{\pm} = \cot \theta + \frac{2(a\mu - a\omega) \sin \theta \cos \theta - (2\omega\ell - \mu\ell) \sin \theta}{\left(\mp \left(\frac{1}{2} + \lambda \right) - m \right) - \ell(2\omega - \mu) \cos \theta + a\omega \sin^2 \theta + a\mu \cos^2 \theta} \quad (3.44)$$

and

$$\begin{aligned}
N_{\pm} = & -\frac{(m \pm \frac{1}{2})^2 + 4 \cos \theta \omega \ell (m \pm \frac{1}{2}) + 4\omega^2 \ell^2}{\sin^2 \theta} \\
& + (\pm \ell(\mu - 2\omega) + (2\omega^2 - \mu^2)2\ell a) \cos \theta + (\pm 2 - a\mu - a\omega)(a\mu - a\omega) \cos^2 \theta \\
& + (4\omega^2 - \mu^2)\ell^2 \mp (\mu - \omega)a - a^2\omega^2 + 2ma\omega + \lambda(\lambda + 1) \\
& + \left(\frac{(\frac{1}{2} \pm m) \cos \theta \pm 2\omega \ell \pm \sin^2 \theta (\ell(\mu - 2\omega) + a(\mu - \omega) \cos \theta)}{(\mp(\frac{1}{2} + \lambda) - m) - \ell(2\omega - \mu) \cos \theta + a\omega \sin^2 \theta + a\mu \cos^2 \theta} \right) \times \\
& (2a(\mu - \omega) \cos \theta - (2\omega - \mu)\ell).
\end{aligned} \tag{3.45}$$

Taking $x = \cos \theta$, equations (3.43) transform into following form:

$$(1 - x^2) \frac{d^2 T_{\pm}}{dx^2} + \bar{M}_{\pm} \frac{dT_{\pm}}{dx} + \bar{N}_{\pm} T_{\pm} = 0 \tag{3.46}$$

where

$$\bar{M}_{\pm} = \frac{\ell(2\omega - \mu)(1 - x^2) - 2a(\mu - \omega)x(1 - x^2)}{(\mp(\frac{1}{2} + \lambda) - m) - \ell(2\omega - \mu)x + a\omega + (a\mu - a\omega)x^2} - 2x \tag{3.47}$$

and

$$\begin{aligned}
\bar{N}_{\pm} = & -\frac{(m \pm \frac{1}{2})^2 + 4\omega \ell (m \pm \frac{1}{2}) x + 4\omega^2 \ell^2}{1 - x^2} \\
& + (\pm \ell(\mu - 2\omega) + (2\omega^2 - \mu^2)2\ell a) x + (\pm 2 - a\mu - a\omega)(a\mu - a\omega) x^2 \\
& + (4\omega^2 - \mu^2)\ell^2 \mp (\mu - \omega)a - a^2\omega^2 + 2ma\omega + \lambda(\lambda + 1) \\
& + \left(\frac{(\frac{1}{2} \pm m)x \pm 2\omega \ell \pm (\ell(\mu - 2\omega) + a(\mu - \omega)x)(1 - x^2)}{(\mp(\frac{1}{2} + \lambda) - m) - \ell(2\omega - \mu)x + a\omega + (a\mu - a\omega)x^2} \right) \times \\
& (2a(\mu - \omega)x - (2\omega - \mu)\ell).
\end{aligned} \tag{3.48}$$

Now we investigate exact solutions to equations (3.46) for some special cases:

i. $\ell = 0$, $a = 0$:

In that case, equations (3.46) take the simple form,

$$\frac{d}{dx} \left((1-x^2) \frac{dT_{\pm}}{dx} \right) + \left(\lambda(\lambda+1) - \frac{(m \pm \frac{1}{2})^2}{1-x^2} \right) T_{\pm} = 0. \quad (3.49)$$

In general, the solutions to those equations are associated Legendre functions $P_{\lambda}^{\nu_{\pm}}$ and $Q_{\lambda}^{\nu_{\pm}}$ of the first and second kind respectively which can be expressed as

$$P_{\lambda}^{\nu_{\pm}} = \frac{1}{\Gamma(1-\nu_{\pm})} \left(\frac{x+1}{x-1} \right)^{\frac{\nu_{\pm}}{2}} F \left(-\lambda, \lambda+1, (1-\nu_{\pm}), \frac{1-x}{2} \right), \quad (3.50)$$

and

$$\begin{aligned} Q_{\lambda}^{\nu_{\pm}} = & A \left(\frac{x-1}{x+1} \right)^{\nu_{\pm}/2} F \left(-\lambda, \lambda+1, \nu_{\pm}+1, \frac{1-x}{2} \right) \\ & + B \left(\frac{x+1}{x-1} \right)^{\nu_{\pm}/2} F \left(-\lambda, \lambda+1, 1-\nu_{\pm}, \frac{1-x}{2} \right) \end{aligned} \quad (3.51)$$

with $\nu_{\pm} \equiv m \pm \frac{1}{2}$ an integer,

$$A = (-1)^{\nu_{\pm}} \frac{\Gamma(-\nu_{\pm})\Gamma(\lambda+\nu_{\pm}+1)}{2\Gamma(1+\lambda-\nu_{\pm})}, B = \frac{(-1)^{\nu_{\pm}}\Gamma(\nu_{\pm})}{2}. \quad (3.52)$$

When λ and ν_{\pm} are integers (ν_{\pm} being even), solutions describe associated Legendre polynomials.

ii. $\ell = 0$, $a \neq 0$ and $\mu = \omega$:

In that case, equations 3.46 can be simplified as

$$\frac{d}{dx} \left((1-x^2) \frac{dT_{\pm}}{dx} \right) + \left(\tilde{\lambda}(\tilde{\lambda}+1) - \frac{(m \pm \frac{1}{2})^2}{1-x^2} \right) T_{\pm} = 0, \quad (3.53)$$

where

$$\tilde{\lambda}(\tilde{\lambda}+1) = \lambda(\lambda+1) + 2ma\omega - a^2\omega^2. \quad (3.54)$$

In that case, the solutions are the same as in the case (i) except that λ is replaced by $\tilde{\lambda}$.

iii. $\ell = 0$, $a \neq 0$, $\mu \neq \omega$:

In that case, equations 3.46 can be written in the following form:

$$\begin{aligned} (1-x^2) \frac{d^2 T_{\pm}}{dx^2} + \left(-2x - \frac{2(a\mu - a\omega)x(1-x^2)}{(\mp(\frac{1}{2} + \lambda) - m) + a\omega + (a\mu - a\omega)x^2} \right) \frac{dT_{\pm}}{dx} \\ + \left(-\frac{(m \pm \frac{1}{2})^2}{1-x^2} + (\pm 2 - a\mu - a\omega)(a\mu - a\omega)x^2 \mp (\mu - \omega)a - a^2\omega^2 \right. \\ \left. + 2ma\omega + \lambda(\lambda + 1) + \frac{(\frac{1}{2} \pm m)x \pm a(\mu - \omega)x(1-x^2)}{(\mp(\frac{1}{2} + \lambda) - m) + a\omega + (a\mu - a\omega)x^2} \right) T_{\pm} = 0. \end{aligned} \quad (3.55)$$

To simplify the equations 3.55, we take the constraints

$$\lambda = \pm a\mu \mp m - \frac{1}{2} \quad (3.56)$$

in the equations satisfied by T_+ and T_- respectively. With these restrictions, equations (3.55) reduce to

$$(1-x^2)^2 \frac{d^2 T_{\pm}}{dx^2} + (Cx^4 + D_{\pm}x^2 + E_{\pm}) T_{\pm} = 0 \quad (3.57)$$

where

$$\begin{aligned} C &= a^2\mu^2 - a^2\omega^2, \\ D_{\pm} &= -a^2\mu^2 + 2a^2\omega^2 \mp m \mp a\omega - \lambda^2 - 2a\omega m - \frac{1}{2}, \\ E_{\pm} &= -\left(m \pm \frac{1}{2}\right)^2 \pm a\omega + \lambda^2 + 2ma\omega - a^2\omega^2 - \frac{1}{2}. \end{aligned} \quad (3.58)$$

Equations of the type (3.57) have exact analytical solutions for $C = 0$, $D_{\pm} = 0$ or $C \neq 0$, $D_{\pm} \neq 0$ [24]. However in our case since $\mu \neq \omega$, $C \neq 0$. In fact, equations (3.57) have simple analytical solutions when $\lambda = \pm \mu a$ (which corresponds to $m = \mp \frac{1}{2}$) which are given by

$$T_{\pm}(x) = d_{\pm} \sin(\sqrt{C}x) + e_{\pm} \cos(\sqrt{C}x) \quad (3.59)$$

for $\mu > \omega$, and

$$T_{\pm}(x) = d_{\pm} e^{(\sqrt{-C}x)} + e_{\pm} e^{(-\sqrt{-C}x)} \quad (3.60)$$

for $\mu < \omega$.

iv. $\ell \neq 0$, $a = 0$ and $\mu = 2\omega$:

In that case, equations 3.46 take the simple form

$$(1-x^2)\frac{d^2T_{\pm}}{dx^2} - 2x\frac{dT_{\pm}}{dx} + \left(\lambda(\lambda+1) - \frac{(m \pm \frac{1}{2})^2 + 4\omega\ell(m \pm \frac{1}{2})x + 4\omega^2\ell^2}{1-x^2} \right) T_{\pm} = 0. \quad (3.61)$$

The equation 3.61 can be arranged as

$$(x^2-1)^2\frac{d^2T_{\pm}}{dx^2} + 2x(x^2-1)\frac{dT_{\pm}}{dx} + (C_{\pm} + D_{\pm}x - \lambda(\lambda+1)x^2) T_{\pm} = 0, \quad (3.62)$$

where

$$C_{\pm} = \lambda(\lambda+1) - \left(m \pm \frac{1}{2}\right)^2 - 4\omega^2\ell^2, \quad D_{\pm} = -4\omega\ell \left(m \pm \frac{1}{2}\right).$$

Under the transformation

$$\xi = \frac{1}{2}(1-x), \quad Y_{\pm} = (x+1)^{-p_{\pm}}(1-x)^{-q_{\pm}}T_{\pm} \quad (3.63)$$

the equation satisfied by Y_{\pm} takes the form

$$\xi(\xi-1)\frac{d^2Y_{\pm}}{d\xi^2} + [(\alpha_{\pm} + \beta_{\pm} + 1) - \gamma_{\pm}]\frac{dY_{\pm}}{d\xi} + \alpha_{\pm}\beta_{\pm}Y_{\pm} = 0 \quad (3.64)$$

whose solution is given by hypergeometric function

$$Y_{\pm}(\xi) = F(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}; \xi). \quad (3.65)$$

Hence the solutions are

$$T_{\pm}(x) = (1+x)^{p_{\pm}}(1-x)^{q_{\pm}}F\left(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}; \frac{1}{2}(1-x)\right). \quad (3.66)$$

Here exponents p_{\pm} and q_{\pm} are obtained by solving the equations,

$$4q_{\pm}(q_{\pm}-1) + 4q_{\pm} - \lambda(\lambda+1) + D_{\pm} + C_{\pm} = 0, \quad (3.67)$$

$$(p_{\pm} - q_{\pm})(2(p_{\pm} + q_{\pm} - 1) + 2) = D_{\pm}, \quad (3.68)$$

whose solutions are given by

$$q_{\pm}^2 = \frac{1}{4} \left(m \pm \frac{1}{2} + 2\omega\ell \right)^2, \quad (3.69)$$

$$p_{\pm}^2 = \frac{1}{4} \left(m \pm \frac{1}{2} - 2\omega\ell \right)^2. \quad (3.70)$$

$\alpha_{\pm}, \beta_{\pm}$ and γ_{\pm} can be obtained from

$$\begin{aligned} \alpha_{\pm} + \beta_{\pm} + 1 &= 2(p_{\pm} + q_{\pm} + 1), \\ \alpha_{\pm}\beta_{\pm} &= (p_{\pm} + q_{\pm})^2 - p_{\pm} - q_{\pm} + 2(p_{\pm} + q_{\pm}) - \lambda(\lambda + 1), \\ \gamma_{\pm} &= 2q_{\pm} + 1. \end{aligned} \quad (3.71)$$

v. $\ell \neq 0$, $a = 0$ and $\mu \neq 2\omega$:

In that case, equations 3.46 reduce to

$$\begin{aligned} (1-x^2) \frac{d^2 T_{\pm}}{dx^2} + \left(-2x + \frac{\ell(2\omega - \mu)(1-x^2)}{(\mp(\frac{1}{2} + \lambda) - m) - \ell(2\omega - \mu)x} \right) \frac{dT_{\pm}}{dx} \\ + \left(-\frac{(m \pm \frac{1}{2})^2 + 4\omega\ell(m \pm \frac{1}{2})x + 4\omega^2\ell^2}{1-x^2} \pm \ell(\mu - 2\omega)x + \lambda(\lambda + 1) \right. \\ \left. + (4\omega^2 - \mu^2)\ell^2 + \frac{(\frac{1}{2} \pm m)x \pm 2\omega\ell \pm \ell(\mu - 2\omega)(1-x^2)}{(\mp(\frac{1}{2} + \lambda) - m) - \ell(2\omega - \mu)x} (\mu - 2\omega)\ell \right) T_{\pm} = 0. \end{aligned} \quad (3.72)$$

As we have done in the case **iii.**, we simplify (3.72) by taking the constraints

$$\mu\ell - 2\omega\ell = \left(\frac{1}{2} + \lambda \right) \pm m \quad (3.73)$$

in the equations satisfied by T_+ and T_- respectively. With these restrictions, equations (3.72) satisfied for T_+ and T_- reduce to

$$(1-x^2)^2 \frac{d^2 T_+}{dx^2} + (1-x)(1-x^2) \frac{dT_+}{dx} + (\bar{C}_+ x^2 + \bar{D}_+ x + \bar{E}_+) T_+ = 0, \quad (3.74)$$

and

$$(1-x^2)^2 \frac{d^2 T_-}{dx^2} - (1+x)(1-x^2) \frac{dT_-}{dx} + (\bar{C}_- x^2 + \bar{D}_- x + \bar{E}_-) T_- = 0, \quad (3.75)$$

respectively, where

$$\begin{aligned}
\bar{C}_{\pm} &= \ell(\mu - 2\omega)(4\omega\ell + 2m \pm 1) - (m \pm \frac{1}{2})^2, \\
\bar{D}_{\pm} &= -4\omega\ell(m \pm 1) - \left(m \pm \frac{1}{2}\right), \\
\bar{E}_{\pm} &= 4\omega\ell(\omega\ell - \mu\ell \pm m) - \left(\frac{1}{2} \pm m\right)(2\mu\ell + 1).
\end{aligned} \tag{3.76}$$

Equations of the type 3.74 and 3.75 seem harder to obtain for exact analytical solutions. However, with $\lambda = \mu\ell$ both equations take the following simple forms:

$$(1+x)^2 \frac{d^2 T_+}{dx^2} + (1+x) \frac{dT_+}{dx} - \left(m + \frac{1}{2}\right)^2 T_+ = 0 \tag{3.77}$$

and

$$(x-1)^2 \frac{d^2 T_-}{dx^2} + (x-1) \frac{dT_-}{dx} - \left(m - \frac{1}{2}\right)^2 T_- = 0, \tag{3.78}$$

whose solutions are given by

$$T_+(x) = c_1(1+x)^{(m+\frac{1}{2})} + c_2(1+x)^{-(m+\frac{1}{2})}, \tag{3.79}$$

$$T_-(x) = d_1(1-x)^{(m-\frac{1}{2})} + d_2(1-x)^{-(m-\frac{1}{2})}, \tag{3.80}$$

where c_1, c_2, d_1 and d_2 are real constants.

vi. $\ell \neq 0, a \neq 0$ and $\mu = 2\omega$:

In that case, equations 3.46 take the following form:

$$\begin{aligned}
&(1-x^2) \frac{d^2 T_{\pm}}{dx^2} - \left(2x + \frac{2a\omega x(1-x^2)}{(\mp(\frac{1}{2} + \lambda) - m) + a\omega + a\omega x^2}\right) \frac{dT_{\pm}}{dx} \\
&+ \left(-\frac{(m \pm \frac{1}{2})^2 + 4\omega\ell(m \pm \frac{1}{2})x + 4\omega^2\ell^2}{1-x^2} + \lambda(\lambda+1) \right. \\
&- 4\ell a\omega^2 x + (\pm 2 - 3a\omega)a\omega x^2 \mp a\omega - a^2\omega^2 + 2ma\omega \\
&\left. + 2a\omega x \frac{(\frac{1}{2} \pm m)x \pm 2\omega\ell \pm a\omega x(1-x^2)}{(\mp(\frac{1}{2} + \lambda) - m) + a\omega + a\omega x^2}\right) T_{\pm} = 0.
\end{aligned} \tag{3.81}$$

Under the constraints

$$2a\omega = m \pm \left(\frac{1}{2} + \lambda\right), \quad (3.82)$$

equations 3.81 can be simplified as

$$(1 - x^2)^2 \frac{d^2 T_+}{dx^2} + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) T_+ = 0, \quad (3.83)$$

and

$$(1 - x^2)^2 \frac{d^2 T_-}{dx^2} + (\bar{a}_0 + \bar{a}_1 x + \bar{a}_2 x^2 + \bar{a}_3 x^3 + \bar{a}_4 x^4) T_- = 0, \quad (3.84)$$

respectively, where

$$\begin{aligned} a_0 &= 3a^2\omega^2 - 4\omega^2\ell^2 - \left(m + \frac{1}{2}\right)(1 + 2a\omega), \\ a_1 &= -4\ell\omega \left(m + \frac{3}{2} + a\omega\right), \\ a_2 &= -6a^2\omega^2 + \left(m + \frac{1}{2}\right)\left(2a\omega - m - \frac{3}{2}\right), \\ a_3 &= 4a\ell\omega^2, \\ a_4 &= 3a^2\omega^2, \end{aligned} \quad (3.85)$$

and

$$\begin{aligned} \bar{a}_0 &= 3a^2\omega^2 - 4\omega^2\ell^2 + \left(m - \frac{1}{2}\right)(1 - 2a\omega), \\ \bar{a}_1 &= -4\ell\omega \left(m + a\omega - \frac{3}{2}\right), \\ \bar{a}_2 &= -6a^2\omega^2 + \left(m - \frac{1}{2}\right)\left(2a\omega - m + \frac{3}{2}\right), \\ \bar{a}_3 &= 4a\ell\omega^2, \\ \bar{a}_4 &= 3a^2\omega^2. \end{aligned} \quad (3.86)$$

In order to obtain an exact analytical solution, equation 3.83 can further be simplified when $\lambda = 2a\omega + 1$ (which corresponds to $m = -\frac{3}{2}$) and $\ell = \frac{1}{2\omega}$

$$(1 + x) \frac{d^2 T_+}{dx^2} + (3a^2\omega^2 + 2a\omega + 3a^2\omega^2 x) T_+ = 0. \quad (3.87)$$

Similarly 3.84 take a simpler form when $\lambda = 2a\omega - 1$ (which corresponds to $m = \frac{3}{2}$) and $\ell = \frac{1}{2\omega}$ as

$$(1-x)\frac{d^2T_-}{dx^2} + (3a^2\omega^2 - 2a\omega - 3a^2\omega^2x)T_- = 0. \quad (3.88)$$

Both equations are of the type

$$(c_2x + b_2)\frac{d^2y}{dx^2} + (c_1x + b_1)\frac{dy}{dx} + (c_0x + b_0)y = 0, \quad (3.89)$$

with $c_1 = 0, b_1 = 0$. Under the transformation [24]

$$y = e^{kx}\Omega(z), \quad z = \frac{1}{\Lambda}(x - \bar{\mu}) \quad (3.90)$$

equation satisfied by $\Omega(z)$ takes the form

$$z\frac{d^2\Omega}{dz^2} + (\bar{b} - z)\frac{d\Omega}{dz} - \bar{a}\Omega = 0, \quad (3.91)$$

where

$$\bar{a} = \frac{b_2k^2 + b_0}{2c_2k}, \quad \bar{b} = 0. \quad (3.92)$$

Here $\bar{\mu} = -\frac{b_2}{c_2}$ and $\Lambda = -\frac{1}{2k}$. k can be calculated from $c_2k^2 + c_0 = 0$. A particular solution of 3.91 (with $\bar{b} = 0$) can be written in terms of confluent hypergeometric function as

$$\Omega(z) = z\Phi(\bar{a} + 1, 2, z). \quad (3.93)$$

In our case, for T_+

$$c_2 = 1, \quad b_2 = 1, \quad c_0 = 3a^2\omega^2, \quad b_0 = 3a^2\omega^2 + 2a\omega. \quad (3.94)$$

So the solution for T_+ can be written as

$$T_+(x) = e^{i\sqrt{3}\omega ax}z\Phi(\bar{a} + 1, 2, z) \quad (3.95)$$

with $z = -2i\sqrt{3}\omega a(x + 1)$ and $\bar{a} = -\frac{i}{\sqrt{3}}$. On the other hand, for T_-

$$c_2 = -1, \quad b_2 = 1, \quad c_0 = -3a^2\omega^2, \quad b_0 = 3a^2\omega^2 - 2a\omega. \quad (3.96)$$

In that case the solution for T_- can be written as

$$T_-(x) = e^{-i\sqrt{3}\omega ax}z\Phi(\bar{a} + 1, 2, z) \quad (3.97)$$

with $z = -2i\sqrt{3}\omega a(x - 1)$ and $\bar{a} = -\frac{i}{\sqrt{3}}$. For both T_+ and T_- , the solutions describe oscillating wave solutions with non-uniform amplitudes.

3.3 Radial equations

Radial equations (3.34), (3.35) can be rearranged as

$$\frac{dR_1}{dr} + \left(\frac{r-M}{\Delta} - i \frac{(ma - \omega(a^2 + r^2 + \ell^2)) 2}{\Delta} \right) R_1 = -\frac{2}{\sqrt{\Delta}} (\mu r + i\lambda) R_2, \quad (3.98)$$

$$\frac{dR_2}{dr} + \left(\frac{r-M}{\Delta} + i \frac{(ma - \omega(a^2 + r^2 + \ell^2)) 2}{\Delta} \right) R_2 = -\frac{2}{\sqrt{\Delta}} (\mu r - i\lambda) R_1. \quad (3.99)$$

It can be seen that R_1 and R_2 satisfy second order differential equations

$$\frac{d^2 R_{\pm}}{dr^2} + P_{\pm} \frac{dR_{\pm}}{dr} + L_{\pm} R_{\pm} = 0, \quad (3.100)$$

where we set $R_1 = R_+$ and $R_2 = R_-$. Here

$$P_{\pm} = \left(\frac{3(r-M)}{\Delta} - \frac{\mu^2 r}{\mu^2 r^2 + \lambda^2} \right) \pm i \left(\frac{\mu \lambda}{\mu^2 r^2 + \lambda^2} \right), \quad (3.101)$$

and

$$L_{\pm} = \text{Re}(L_{\pm}) + i \text{Im}(L_{\pm}), \quad (3.102)$$

where

$$\begin{aligned} \text{Re}(L_{\pm}) &= \frac{1}{\Delta} + \frac{4}{\Delta^2} (ma - \omega(a^2 + r^2 + \ell^2))^2 - \frac{4}{\Delta} (\mu^2 r^2 + \lambda^2) \\ &- \frac{(r-M)\mu^2 r}{\Delta(\mu^2 r^2 + \lambda^2)} + \frac{2\mu\lambda}{\Delta(\mu^2 r^2 + \lambda^2)} (ma - \omega(a^2 + r^2 + \ell^2)), \end{aligned} \quad (3.103)$$

and

$$\begin{aligned} \text{Im}(L_{\pm}) &= \mp \left(\frac{2\mu^2}{\Delta(\mu^2 r^2 + \lambda^2) - \frac{6(r-M)}{\Delta^2}} \right) (ma - \omega(a^2 + r^2 + \ell^2)) \\ &\mp \frac{1}{\Delta} \left(\frac{r-M}{\mu^2 r^2 + \lambda^2} - 4\omega r \right). \end{aligned} \quad (3.104)$$

To get the radial equations in the form of a wave equation, we follow the method applied in Chandrasekhar's book [10] but with $\ell \neq 0$. Hence we consider the transformations

$$P_1 = i\Delta^{1/2} R_1, \quad P_2 = \Delta^{1/2} R_2. \quad (3.105)$$

Let $\Omega = r^2 + a^2 + \ell^2 - \frac{ma}{\omega}$. With these transformations, equations (3.98) and (3.99) take the form

$$\frac{dP_1}{dr} + 2i\frac{\omega\Omega}{\Delta}P_1 = \frac{-2i(\mu r + i\lambda)}{\Delta^{1/2}}P_2, \quad (3.106)$$

$$\frac{dP_2}{dr} - 2i\frac{\omega\Omega}{\Delta}P_2 = \frac{2i(\mu r - i\lambda)}{\Delta^{1/2}}P_1. \quad (3.107)$$

Let

$$\frac{du}{dr} = \frac{\Omega}{\Delta}, \quad \beta^2 = M^2 - a^2 + \ell^2. \quad (3.108)$$

Then in terms of the new independent variable u , we obtain

$$\frac{dP_1}{du} + 2i\omega P_1 = \frac{-2i(\mu r + i\lambda)}{\Omega}\Delta^{1/2}P_2, \quad (3.109)$$

$$\frac{dP_2}{du} - 2i\omega P_2 = \frac{2i(\mu r - i\lambda)}{\Omega}\Delta^{1/2}P_1, \quad (3.110)$$

where

$$u = r + \frac{2Mr_+ + 2\ell^2 - \frac{ma}{\omega}}{2\beta} \ln\left(\frac{r - r_+}{r_+}\right) - \frac{2Mr_- + 2\ell^2 - \frac{ma}{\omega}}{2\beta} \ln\left(\frac{r - r_-}{r_-}\right), \quad (r > r_+) \quad (3.111)$$

for $M^2 > a^2 - \ell^2$. Here $r_+ = M + \beta$ and $r_- = M - \beta$. The relation (3.111) is single-valued for $r > r_+$ if the inequality

$$r_+^2 + a^2 + \ell^2 - \frac{ma}{\omega} > 0 \quad (3.112)$$

is satisfied. This requires that in the frequency range

$$\omega_2 < \omega < \omega_1, \quad (3.113)$$

where

$$\omega_2 = \frac{ma}{2Mr_+ + 2\ell^2}, \quad \omega_1 = \frac{ma}{\ell^2 + a^2}, \quad m > 0, \quad (3.114)$$

u becomes single-valued in r in the region where $r > r_+$. It is seen that as $r \rightarrow \infty$, $u \rightarrow \infty$ and as $r \rightarrow (r_+)^+$, $u \rightarrow -\infty$. For completeness, we also note that for the critical case when $M^2 = a^2 - \ell^2$, u becomes

$$u = r - \frac{(2\ell^2 + 2M^2 - \frac{ma}{\omega})}{r - M} + 2M \ln(r - M), \quad (r > M). \quad (3.115)$$

Concentrating on the case $M^2 > a^2 - \ell^2$, let us make another transformation

$$\vartheta = \arctan\left(\frac{\mu r}{\lambda}\right). \quad (3.116)$$

With the new definitions

$$P_1 = \phi_1 e^{-\frac{1}{2}i\vartheta}, \quad P_2 = \phi_2 e^{\frac{1}{2}i\vartheta}, \quad (3.117)$$

equations (3.109), (3.110) take the forms

$$\frac{d\phi_1}{du} + 2i\omega \left(1 - \frac{\lambda\mu\Delta}{4\omega(\lambda^2 + \mu^2 r^2)\Omega}\right) \phi_1 = 2\frac{\sqrt{\lambda^2 + \mu^2 r^2}}{\Omega} \Delta^{1/2} \phi_2, \quad (3.118)$$

$$\frac{d\phi_2}{du} - 2i\omega \left(1 - \frac{\lambda\mu\Delta}{4\omega(\lambda^2 + \mu^2 r^2)\Omega}\right) \phi_2 = 2\frac{\sqrt{\lambda^2 + \mu^2 r^2}}{\Omega} \Delta^{1/2} \phi_1. \quad (3.119)$$

Redefining the independent variable as

$$\hat{u} = u - \frac{1}{4\omega} \arctan\left(\frac{\mu r}{\lambda}\right), \quad (3.120)$$

equations (3.118) and (3.119) can be simplified as

$$\frac{d\phi_1}{d\hat{u}} + 2i\omega\phi_1 = W\phi_2, \quad (3.121)$$

$$\frac{d\phi_2}{d\hat{u}} - 2i\omega\phi_2 = W\phi_1, \quad (3.122)$$

where

$$W = \frac{2(\lambda^2 + \mu^2 r^2)^{3/2} \Delta^{1/2}}{(\lambda^2 + \mu^2 r^2)\Omega - \frac{\lambda\mu\Delta}{4\omega}}. \quad (3.123)$$

By further defining $Z_1 = \phi_1 + \phi_2$ and $Z_2 = \phi_1 - \phi_2$, equations (3.121) and (3.122) can be rewritten

$$\frac{dZ_1}{d\hat{u}} - WZ_1 = -2i\omega Z_2, \quad (3.124)$$

$$\frac{dZ_2}{d\hat{u}} + WZ_2 = -2i\omega Z_1. \quad (3.125)$$

From equations (3.124) and (3.125) we obtain one-dimensional wave equations

$$\frac{d^2 Z_1}{d\hat{u}^2} + 4\omega^2 Z_1 = V_+ Z_1, \quad (3.126)$$

$$\frac{d^2 Z_2}{d\hat{u}^2} + 4\omega^2 Z_2 = V_- Z_2, \quad (3.127)$$

where the effective potentials

$$V_{\pm} = W^2 \pm \frac{dW}{d\hat{u}}. \quad (3.128)$$

We calculate the potentials as

$$\begin{aligned} V_{\pm}(r) = & \frac{2(\lambda^2 + \mu^2 r^2)^{3/2} \Delta^{1/2}}{I^2} \left[2(\lambda^2 + \mu^2 r^2)^{3/2} \Delta^{1/2} \right. \\ & \left. \pm 3\mu^2 r \Delta \pm (\lambda^2 + \mu^2 r^2)(r - M) \right] \\ \mp & \frac{\Delta(\lambda^2 + \mu^2 r^2)}{I} \left(2\mu^2 \Omega r + 2(\lambda^2 + \mu^2 r^2)r - \frac{\lambda\mu(r - M)}{2\omega} \right) \Bigg], \end{aligned} \quad (3.129)$$

where

$$I = (\lambda^2 + \mu^2 r^2) \Omega - \frac{\lambda\mu\Delta}{4\omega}. \quad (3.130)$$

We see that, the effective potentials depend on gravitomagnetic monopole moment ℓ via the functions Δ and Ω , where $\ell = 0$ case is discussed in [10]. We also report that, for $\mu = 0$, the potentials take the simple form

$$V_{\pm}(r) = \frac{2\Delta^{1/2}\lambda}{\Omega^2} \left\{ 2\lambda\Delta^{1/2} \pm (r - M) \mp \frac{2\Delta r}{\Omega} \right\}. \quad (3.131)$$

4 Discussion

Potentials become singular when $I = 0$. Moreover they possess local extremums when $\frac{dV_{\pm}}{dr} = 0$ which can also be seen in the following graphs. To understand the behaviour of the potentials V_{\pm} in the physical region $r > r_+$, we plot them as a function of radial distance r for massive and massless particles. We take the physical parameters $M = 1$, $\lambda = 1$, $m = 0.5$, $a = 0.95$.

Here, Figure 1 and 2 describe the effective potentials V_{\pm} for massive particle with $\mu = 1$. In the first graph, frequency ω is kept constant ($\omega = 0.2$). We examine the potentials by changing gravitomagnetic monopole moment ℓ . We see that, for sufficiently small values of ℓ including $\ell = 0$, potentials have a local maximum in the physical region $r > r_+$. It is also observed that while ℓ increases, local maxima of the potentials seems to disappear. Potentials become bounded regardless of the value of ℓ and approach a constant value in the sufficiently large values of r (or $r \rightarrow \infty$). In the second graph, we

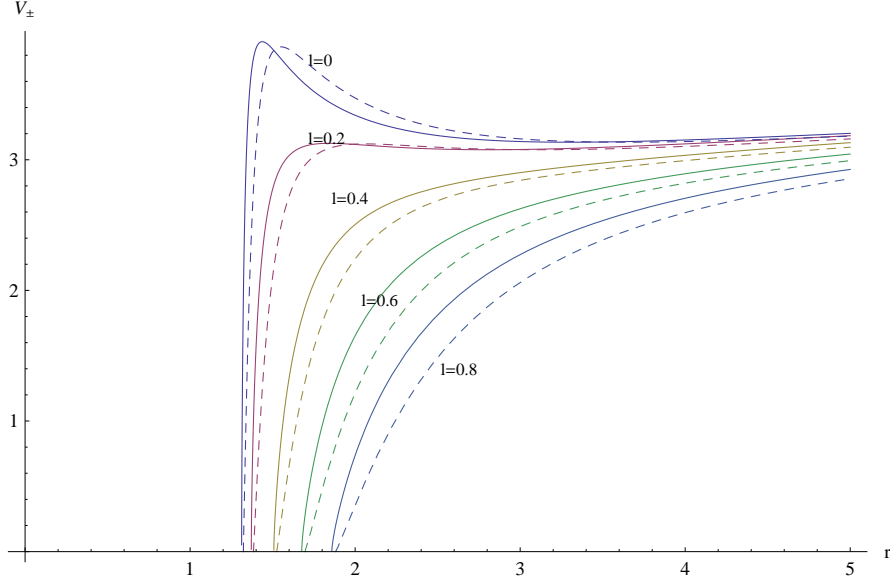


Figure 1: Graph of V_{\pm} with $M = 1$, $\lambda = 1$, $\mu = 1$, $\omega = 0.2$, $m = 0.5$, $a = 0.95$

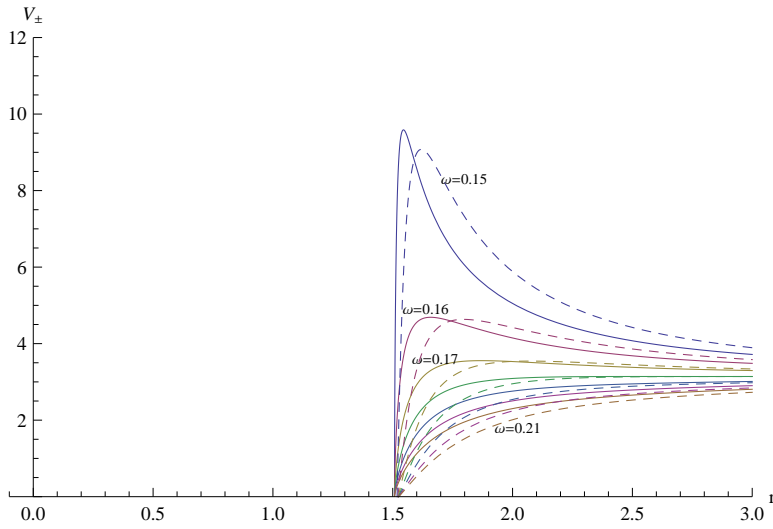


Figure 2: Graph of V_{\pm} with $M = 1$, $\lambda = 1$, $\mu = 1$, $\ell = 0.4$, $m = 0.5$, $a = 0.95$

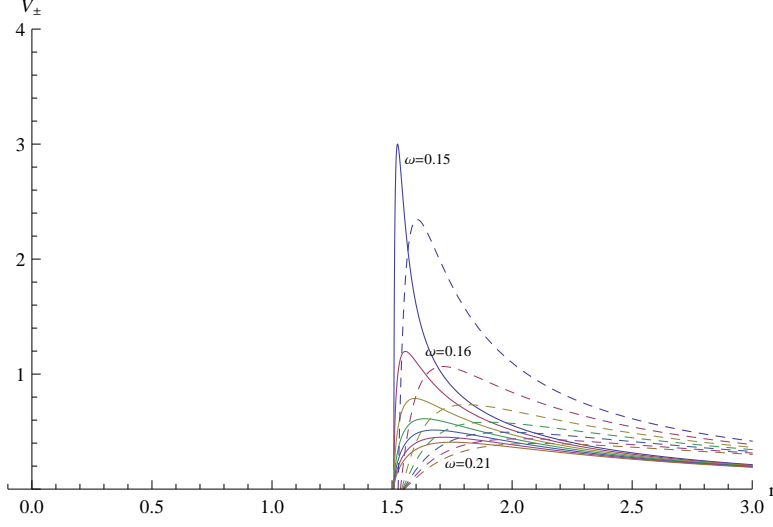


Figure 3: Graph of V_{\pm} with $M = 1$, $\lambda = 1$, $\mu = 0$, $\ell = 0.4$, $m = 0.5$, $a = 0.95$

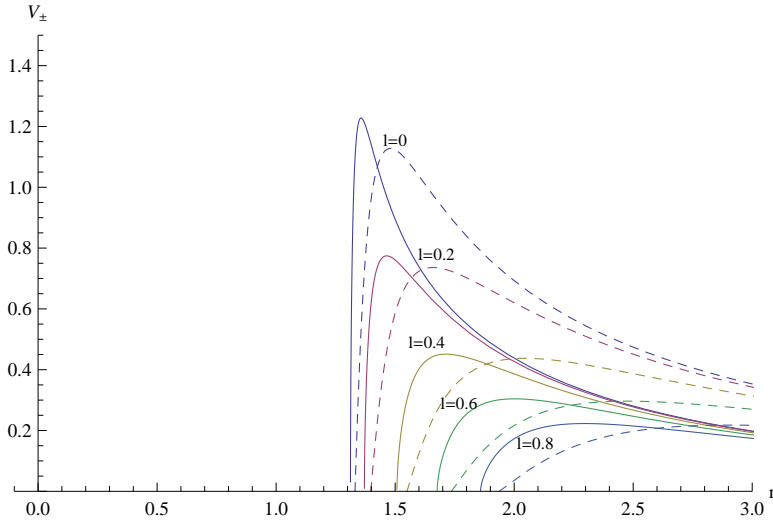


Figure 4: Graph of V_{\pm} with $M = 1$, $\lambda = 1$, $\mu = 0$, $\omega = 0.2$, $m = 0.5$, $a = 0.95$

keep gravitomagnetic monopole moment ℓ fixed ($\ell = 0.4$). In that case, we investigate the behaviour of the potentials by changing ω that can take values in the range (3.113) for \hat{u} or u to be single-valued. We see that potentials have some local maxima in low frequencies. While the frequency increases, local extremums again disappear as in Figure 1 and potentials behave similarly in the sufficiently large distances.

In Figures 3 and 4, we plot the potentials for massless ($\mu = 0$) spin- $\frac{1}{2}$ particles. We take same values for the physical parameters. As in the previous case, in one plot ω is fixed while ℓ is changed, in the other ℓ is kept constant while ω is changed. It can be seen that, contrary to Figures 1 and 2 plotted for massive particle, potentials have local maxima for all values of ω and ℓ . As expected, maximum value of the potentials in the physical range $r > r_+$ decreases when compared to potentials in Figures 1 and 2.

5 Conclusion

In this work, we examine the Dirac equation in 4-dimensional Kerr-Taub-NUT spacetime described by the physical parameters, the mass M , angular speed a per unit mass and gravitomagnetic monopole moment ℓ . In the absence of gravitomagnetic monopole moment ℓ , equations are separable in the Boyer-Lindquist coordinates as shown in [9, 10]. We show that, in the same coordinate system, equations are still separable in the presence of gravitomagnetic monopole moment ℓ . We obtain angular and radial equations for arbitrary ℓ . We find some exact solutions to the angular equations with and without gravitomagnetic monopole moment ℓ and rotation parameter a . We see that when the mass of the particle is equal to or twice the frequency of the spinor wave function, some angular solutions can be represented in terms of hypergeometric functions.

We also discuss the radial equations and get a wave equation with an effective potential. To realize the physical interpretations of the potentials, we plot them as a function of radial distance r . From the plots, it can be seen that the strength of the potential barrier decreases while NUT charge ℓ increases. We believe that to better understand the physical significance of the NUT charge, Dirac Hamiltonian should be constructed for the Dirac equation [23]. This will be the subject of our future research.

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